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Optimal quadrature formulas with positive coefficients in $L_2^{(m)}(0,1)$ space

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ABSTRACT

In the Sobolev space $L_2^{(m)}(0,1)$ optimal quadrature formulas of the form $\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^N C_\beta \varphi(x_\beta)$ with the nodes $x_i = \eta_i h, x_{N-i} = 1 - \eta_i h, i = \overline{0, t-1}, 0 \leq \eta_0 < \eta_1 < \cdots < \eta_{t-1} < t, t \in \mathbb{N}, x_\beta = h\beta, t \leq \beta \leq N-t, h = \frac{1}{N}$ are investigated. For optimal coefficients C_β explicit forms are obtained and the norm of the error functional is calculated for any natural numbers m and N. In particular, in the case $t = 1$ and $\eta_0 = 0.205$ for $m = 2, 3, \ldots, 14$ optimal quadrature formulas with positive coefficients are numerically obtained and some of them are compared with well-known optimal formulas.

1. Introduction. Statement of the problem

It is known that numerical integration formulas or quadrature formulas are methods for the approximate evaluation of definite integrals. They are needed for the computation of those integrals for which either the antiderivative of the integrand cannot be expressed in terms of elementary functions or for which the integrand is available only at discrete points, for example from experimental data. In addition and even more important, quadrature formulas provide a basic and important tool for the numerical solution of differential and integral equations.

There are various methods in the theory of quadrature, which allow us to approximately calculate integrals with the help of a finite number of values of the integrand. The present paper is also devoted to one of the methods, i.e. to the construction of optimal quadrature formulas for approximate evaluation of definite integrals in the space $L_2^{(m)}(0,1)$, equipped with the norm

$$\|\varphi(x)\|_{L_2^{(m)}(0,1)} = \left\{ \int_0^1 (\varphi^{(m)}(x))^2 dx \right\}^{1/2}.$$

Consider a quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^N C_\beta \varphi(x_\beta) \tag{1.1}$$

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^N C_\beta \delta(x - x_\beta) \tag{1.2}$$

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in the Sobolev space $L_2^{(m)}(0,1)$. Here C_β and x_β are the coefficients and the nodes of the quadrature formula (1.1), respectively, $\delta(x)$ is the Dirac delta function, $\varepsilon_{[0,1]}(x)$ is the indicator of the interval [0, 1].

In order that the error functional (1.2) is defined on the space $L_2^{(m)}(0,1)$ it is necessary to impose the following conditions (see [1])

$$(\ell(x), x^\alpha) = 0, \quad \alpha = 0, 1, 2, \dots, m-1. \tag{1.3}$$

The difference

$$(\ell(x), \varphi(x)) = \int_0^1 \varphi(x)\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)\mathrm{d}x \tag{1.4}$$

is called *the error* of the quadrature formula (1.1).

By the Cauchy–Schwarz inequality

$$|(\ell(x), \varphi(x))| \le \|\varphi(x)|L_2^{(m)}(0,1)\| \cdot \|\ell(x)|L_2^{(m)*}(0,1)\|$$

the error (1.4) of the formula (1.1) is estimated by the norm

$$\left\|\ell(x)|L_2^{(m)*}\right\| = \sup_{\left\|\varphi(x)|L_2^{(m)}\right\|=1} |(\ell(x), \varphi(x))|$$

of the error functional (1.2). Here $L_2^{(m)*}(0,1)$ is the conjugate space to the space $L_2^{(m)}(0,1)$. Consequently, estimation of the error (1.4) of the quadrature formula (1.1) on functions of the space $L_2^{(m)}(0,1)$ is reduced to finding the norm of the error functional $\ell(x)$ in the conjugate space $L_2^{(m)*}(0,1)$.

Clearly, that norm of the error functional $\ell(x)$ depends on the coefficients C_β and the nodes x_β. The problem of finding the minimum of the norm of the error functional $\ell(x)$ by coefficients C_β and nodes x_β, is called the *Nikolskii problem*, and the obtained formula is called *optimal quadrature formula in the sense of Nikolskii*. This problem was first considered in [2]. This problem was further investigated by many authors for various cases (see e.g. [3–8] and the references therein). Minimization of the norm of the error functional $\ell(x)$ by coefficients C_β when the nodes are fixed is called *Sard's problem*. And the obtained formula is called *optimal quadrature formula in the sense of Sard*. This problem was first investigated in [9].

There are several methods for the construction of optimal quadrature formulas in the sense of Sard such as the spline method, the φ-function method (see e.g. [10,3]) and Sobolev's method. Note that the Sobolev method is based on the construction of a discrete analogue of a linear differential operator (see e.g. [11]). In the different spaces, based on these methods, the Sard problem was investigated by many authors (see, for example, [13,3,5,14–24,10,25,26,11,1,27–29] and the references therein).

Furthermore, explicit formulas for coefficients of optimal quadrature formulas for any m and for any number $N+1$ of the nodes x_β in the space $L_2^{(m)}$ when the nodes are equally spaced were obtained in the works [17,21,23,29].

Schoenberg and Silliman [26] investigated the Sard problem for the case $N \to \infty$ in the space $L_2^{(m)}$. In [26] an algorithm for finding the optimal coefficients was given with the help of a spline of degree $2m-1$. In the cases $m = 2, 3, \dots, 7$ the coefficients are calculated using a Computer. It was especially noted that among optimal coefficients the negative coefficient $B_4^{(7)}$ appears when $m = 7$. It appears that by increasing m the number of negative coefficients increases. This is confirmed by computations of the optimal coefficients for $m \le 30$ in [28].

It is known that the optimal quadrature formulas with positive coefficients play a very important role in applications.

There is a question: can we obtain in some way the optimal quadrature formulas with positive coefficients in the Sobolev space $L_2^{(m)}(0,1)$?

The main objective of the present paper is to construct optimal quadrature formulas in the sense of Sard in the space $L_2^{(m)}(0,1)$ with the nodes

$$\begin{aligned} &x_i = \eta_i h, \qquad x_{N-i} = 1 - \eta_i h, \quad i = \overline{0, t-1},\ 0 \le \eta_0 < \eta_1 < \cdots < \eta_{t-1} < t,\ t \in \mathbb{N}, \\ &x_\beta = h\beta, \quad t \le \beta \le N-t, \qquad h = \frac{1}{N}, \quad t = \begin{cases} \dfrac{m}{2} & \text{when } m \text{ is even}, \\ \left[\dfrac{m}{2}\right] + 1 & \text{when } m \text{ is odd}, \end{cases} \end{aligned} \tag{1.5}$$

using the Sobolev method and numerically choosing the parameters η_i, $i = \overline{0, t-1}$ to obtain the optimal quadrature formulas of the form (1.1) with positive coefficients. Here $[a]$ is the integer part of the number a. This means to find the coefficients C_β which satisfy the following equality

$$\left\|\mathring{\ell}(x)|L_2^{(m)*}\right\| = \inf_{C_\beta} \left\|\ell(x)|L_2^{(m)*}\right\| \tag{1.6}$$

for the nodes (1.5).

Thus, in order to construct of optimal quadrature formula in the sense of Sard with the nodes (1.5) in the space $L_2^{(m)}(0, 1)$ we need consequently to solve the following problems.

Problem 1. Find the norm of the error functional $\ell(x)$ of quadrature formulas (1.1) in the space $L_2^{(m)*}(0, 1)$.

Problem 2. Find the coefficients C_β which satisfy equality (1.6) with the nodes (1.5).

It is known [11,1,27] that square of the norm of the error functional (1.2) for arbitrary fixed x_β has the following form

$$\left\|\ell(x)|L_2^{(m)*}(0,1)\right\|^2 = (-1)^m\left(\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma \frac{|x_\beta - x_\gamma|^{2m-1}}{2\cdot(2m-1)!} - 2\sum_{\beta=0}^{N} C_\beta \frac{x_\beta^{2m} + (1-x_\beta)^{2m}}{2\cdot(2m)!} + \frac{1}{(2m+1)!}\right). \tag{1.7}$$

Obviously, that square of the norm (1.7) of the error functional $\ell(x)$ is a multidimensional function with respect to the coefficients C_β. Moreover, the error functional $\ell(x)$ satisfies the conditions (1.3). In [1,27] taking these facts into account the Lagrange function with conditions (1.3) is constructed for finding the *conditional minimum* of (1.7) and differentiating the Lagrange function by C_β and λ_α (where λ_α are Lagrange factors) the system of linear equations

$$\sum_{\gamma=0}^{N} C_\gamma \frac{|x_\beta - x_\gamma|^{2m-1}}{2\cdot(2m-1)!} + \sum_{\alpha=0}^{m-1}\lambda_\alpha x_\beta^\alpha = \frac{x_\beta^{2m} + (1-x_\beta)^{2m}}{2\,(2m)!}, \quad x_\beta \in [0,1], \tag{1.8}$$

$$\sum_{\gamma=0}^{N} C_\gamma x_\gamma^\alpha = \frac{1}{\alpha+1}, \quad \alpha = \overline{0, m-1} \tag{1.9}$$

was obtained.

It was proved in [1,27] that this system has a unique solution and this solution gives a minimum to the expression (1.7). This means that the square of the norm of the error functional $\ell(x)$ being a quadratic function of the coefficients C_β has a unique minimum in the concrete value of $C_\beta = \overset{\circ}{C}_\beta$. The quadrature formula with the coefficients $\overset{\circ}{C}_\beta$, when the nodes are fixed, is called *optimal quadrature formula in the sense of Sard* and the coefficients $\overset{\circ}{C}_\beta$ are called *optimal*.

Below for convenience the optimal coefficients $\overset{\circ}{C}_\beta$ remain as C_β.

Thus, Problem 1 is already solved by Sobolev in the space $L_2^{(m)}$ and Problem 2 is reduced to the system of linear equations for optimal coefficients. It should be noted that in [1,27] Problem 1 is solved for the multidimensional case, i.e. for cubature formulas.

In the present paper we will solve system (1.8), (1.9), i.e. we will find explicit forms of the optimal coefficients C_β and will calculate the norm of the optimal error functional $\ell(x)$ of formula (1.1) with the nodes (1.5) for any natural numbers m and N, $N \geq m$.

2. Definitions and known formulas

In this section we give some definitions and formulas which are necessary in the proof of the main results.

The *Euler–Frobenius polynomials* $E_k(x)$, $k = 1, 2, \ldots$ are defined by the following formula (see, e.g. [11,12])

$$E_k(x) = \frac{(1-x)^{k+2}}{x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \frac{x}{(1-x)^2}, \tag{2.1}$$

$E_0(x) = 1$.

For the Euler–Frobenius polynomials the identity

$$E_k(x) = x^k E_k\left(\frac{1}{x}\right), \tag{2.2}$$

holds and also the following theorem is true.

Theorem 2.1 (*Lemma 3 of [22]*). *The polynomial $P_k(x)$ which is determined by the formula*

$$P_k(x) = (x-1)^{k+1}\sum_{i=0}^{k+1}\frac{\Delta^i 0^{k+1}}{(x-1)^i} \tag{2.3}$$

is the Euler–Frobenius polynomial (2.1) of degree k, i.e. $P_k(x) = E_k(x)$, where $\Delta^i 0^k = \sum_{l=1}^{i}(-1)^{i-l}C_i^l l^k$.

The following formula is valid [30]:

$$\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i \gamma^k|_{\gamma=n}, \tag{2.4}$$

where $\Delta^i\gamma^k$ is the finite difference of order i of γ^k.

Lastly we give the following well-known formulas from [31]

$$\sum_{\gamma=0}^{\beta-1}\gamma^k=\sum_{j=1}^{k+1}\frac{k!\,B_{k+1-j}}{j!\,(k+1-j)!}\,\beta^j, \tag{2.5}$$

where B_{k+1-j} are Bernoulli numbers,

$$\Delta^\alpha x^\nu=\sum_{p=0}^{\nu}\binom{\nu}{p}\,\Delta^\alpha 0^p x^{\nu-p}. \tag{2.6}$$

3. Auxiliary results

3.1. Lemmas

In the proofs of the main results we need the following lemmas.

Lemma 3.1. *The optimal coefficients of quadrature formulas of the form (1.1) with the error functional (1.2) and the nodes (1.5) in the space $L_2^{(m)}(0,1)$ have the following representation*

$$C_\beta=h\left(1+\sum_{k=1}^{m-1}d_k\left(q_k^\beta+q_k^{N-\beta}\right)\right) \tag{3.1}$$

when $\beta=t,t+1,\dots,N-t$. Here d_k are unknown parameters, q_k are roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$ and $|q_k|<1$.

Lemma 3.2. *The identity*

$$\sum_{i=1}^{\alpha}\frac{-q_k^{N-s+1}+(-1)^i q_k^{s+i}}{(q_k-1)^{i+1}}\Delta^i 0^\alpha=(-1)^\alpha\sum_{i=1}^{\alpha}\frac{q_k^{s+1}+(-1)^{i+1}q_k^{N-s+i}}{(q_k-1)^{i+1}}\Delta^i 0^\alpha \tag{3.2}$$

holds, where $\alpha=1,2,\dots,m-1,\ s=0,1,\dots,q_k$ are roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$.

Lemma 3.3. *The coefficients of optimal quadrature formulas of the form (1.1) with the error functional (1.2) and the nodes (1.5) in the space $L_2^{(m)}(0,1)$ satisfy the system*

$$\sum_{\beta=0}^{t-1}C_\beta\eta_\beta^\alpha=h\left(\sum_{\beta=1}^{t-1}\beta^\alpha+\frac{0^\alpha}{2}+\sum_{k=1}^{m-1}d_k\sum_{i=0}^{\alpha}\frac{(-1)^i q_k^{t+i}-q_k^{N-t+1}}{(q_k-1)^{i+1}}\Delta^i t^\alpha\right), \tag{3.3}$$

here $\alpha=\begin{cases}0,2,4,\dots,m-2 \text{ when } m-\text{even},\\ 0,2,4,\dots,m-1 \text{ when } m-\text{odd},\end{cases}$ $0^\alpha=\begin{cases}1, & \alpha=0,\\ 0, & \alpha\neq 0,\end{cases}$

$t=\begin{cases}\frac{m}{2} & \text{when } m \text{ is even},\\ \left[\frac{m}{2}\right]+1 & \text{when } m \text{ is odd},\end{cases}$ *where $[a]$ is the integer part of the number a, d_k are unknown parameters, q_k are roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$, $|q_k|<1$.*

Lemma 3.4. *Let $m>n$ and $m,n\in\mathbb{N}$ then the following formula for binomial coefficients is true*

$$C_m^{m-1}C_{m-1}^n-C_m^{m-2}C_{m-2}^n+\cdots+(-1)^{m-n-2}C_m^{n+1}C_{n+1}^n+(-1)^{m-n-1}C_m^nC_n^n=C_m^n. \tag{3.4}$$

Lemma 3.5 *([1]). The identity*

$$y^k=\sum_{i=1}^{k}\frac{y^{[i]}}{i!}\Delta^i 0^k \tag{3.5}$$

is true, where $y^{[k]}=y(y-1)(y-2)\cdots(y-k+1)$, $\Delta^i 0^k=\sum_{l=1}^{i}(-1)^{i-l}C_i^l l^k$.

3.2. Proofs of lemmas

The proof of Lemma 3.1 is obtained similarly as the proof of Theorem 3 of the work [22] for $\beta=\overline{t,N-t}$.

Proof of Lemma 3.2. Denote the left and the right hand sides of the identity (3.2) by A_1 and A_2, respectively, i.e.

$$A_1=\sum_{i=1}^{\alpha}\frac{-q_k^{N-s+1}+(-1)^i q_k^{s+i}}{(q_k-1)^{i+1}}\Delta^i 0^\alpha,\qquad A_2=(-1)^\alpha\sum_{i=1}^{\alpha}\frac{q_k^{s+1}+(-1)^{i+1}q_k^{N-s+i}}{(q_k-1)^{i+1}}\Delta^i 0^\alpha.$$

Then, using Theorem 2.1 and the identity (2.2), for A_1 we obtain

$$A_1=\frac{-q_k^{N-s+1}+(-1)^\alpha q_k^{s+1}}{(q_k-1)^{\alpha+1}}E_{\alpha-1}(q_k). \tag{3.6}$$

For A_2 also, using Theorem 2.1 and the identity (2.2), we have

$$A_2=\frac{-q_k^{N-s+1}+(-1)^\alpha q_k^{s+1}}{(q_k-1)^{\alpha+1}}E_{\alpha-1}(q_k). \tag{3.7}$$

From equalities (3.6) and (3.7) we obtain the statement of Lemma 3.2.
Lemma 3.2 is proved. □

Proof of Lemma 3.3. Here we use Lemma 3.1 and equality (1.9). The optimal coefficients C_β and the nodes x_β have to satisfy equality (1.9), i.e.

$$\sum_{\beta=0}^{N}C_\beta x_\beta^\alpha=\frac{1}{\alpha+1},\quad \alpha=0,1,2,\ldots,m-1. \tag{3.8}$$

From symmetry of the nodes (1.5) it follows that

$$C_\beta=C_{N-\beta}. \tag{3.9}$$

Here we consider the case $\alpha=0$ separately.
Suppose $\alpha=0$ then from (3.8) we have

$$\sum_{\beta=0}^{N}C_\beta=1. \tag{3.10}$$

Taking into account (3.9) and using Lemma 3.1 for the left side of (3.10) we get

$$\begin{aligned}\sum_{\beta=0}^{N}C_\beta&=2\sum_{\beta=0}^{t-1}C_\beta+\sum_{\beta=t}^{N-t}h\left(1+\sum_{k=1}^{m-1}d_k(q_k^\beta+q_k^{N-\beta})\right)\\&=2\sum_{\beta=0}^{t-1}C_\beta-h\left(2t-1-\sum_{k=1}^{m-1}d_k\frac{q_k^t-q_k^{N-t+1}}{1-q_k}\right)+1.\end{aligned}$$

Hence, considering (3.10), we obtain

$$\sum_{\beta=0}^{t-1}C_\beta=h\left(\frac{2t-1}{2}-\sum_{k=1}^{m-1}d_k\frac{q_k^t-q_k^{N-t+1}}{1-q_k}\right). \tag{3.11}$$

This is the first equation of system (3.3) corresponding to the case $\alpha=0$. Thus we have proved the lemma for the case $\alpha=0$.

Suppose $\alpha=1,2,\ldots,m-1$. Then, using the symmetry of the nodes (1.5) and keeping in mind equality (3.9), for the left side of (3.8) we obtain

$$\sum_{\beta=0}^{N}C_\beta x_\beta^\alpha=\sum_{\beta=0}^{t-1}C_\beta\left(x_\beta^\alpha+(1-x_\beta)^\alpha\right)+\sum_{\beta=t}^{N-t}C_\beta x_\beta^\alpha=Y_1+Y_2,$$

where

$$Y_1=\sum_{\beta=0}^{t-1}C_\beta\left(x_\beta^\alpha+(1-x_\beta)^\alpha\right),\qquad Y_2=\sum_{\beta=t}^{N-t}C_\beta x_\beta^\alpha.$$

Substituting the expression (3.1) of the optimal coefficients C_β into Y_2, adding and subtracting the expressions

$$\sum_{\beta=1}^{t-1}h\left(1+\sum_{k=1}^{m-1}d_k\left(q_k^\beta+q_k^{N-\beta}\right)\right)(h\beta)^\alpha$$

and

$$\sum_{\beta=N-t+1}^{N-1} h\left(1+\sum_{k=1}^{m-1} d_k\left(q_k^{\beta}+q_k^{N-\beta}\right)\right)(h\beta)^{\alpha},$$

we have

$$Y_2 = h^{\alpha+1}\left(\sum_{\beta=1}^{N-1}\beta^{\alpha}+\sum_{k=1}^{m-1} d_k\left(\sum_{\beta=1}^{N-1} q_k^{\beta}\beta^{\alpha}+\sum_{\beta=1}^{N-1} q_k^{N-\beta}\beta^{\alpha}\right)\right)$$
$$-h\sum_{\beta=1}^{t-1}\left((h\beta)^{\alpha}+(1-h\beta)^{\alpha}\right)\left(1+\sum_{k=1}^{m-1} d_k\left(q_k^{\beta}+q_k^{N-\beta}\right)\right).$$

Using equalities (2.4)–(2.6) and the binomial formula, then grouping in powers of h and taking into account Lemma 3.2, for Y_2 we have

$$Y_2 = h^{\alpha+1}\left[\sum_{k=1}^{m-1} d_k\sum_{i=1}^{\alpha}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^{\alpha}-\sum_{\beta=1}^{t-1}\beta^{\alpha}\left(1+\sum_{k=1}^{m-1} d_k\left(q_k^{\beta}+q_k^{N-\beta}\right)\right)\right](1+(-1)^{\alpha})$$
$$+\sum_{j=1}^{\alpha-1}\frac{\alpha! h^{j+1}}{j!(\alpha-j)!}\left[\frac{B_{j+1}}{j+1}+\sum_{k=1}^{m-1} d_k\sum_{i=1}^{j}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^j+(-1)^{j+1}\sum_{\beta=1}^{t-1}\beta^j\left(1+\sum_{k=1}^{m-1} d_k\left(q_k^{\beta}+q_k^{N-\beta}\right)\right)\right]$$
$$-h\left(\frac{2t-1}{2}-\sum_{k=1}^{m-1} d_k\frac{q_k^t-q_k^{N-t+1}}{1-q_k}\right)+\frac{1}{\alpha+1}.$$

Now we consider Y_1. When $\beta=\overline{0,t-1}$ then from (1.5) the nodes $x_{\beta}=\eta_{\beta}h$ and applying the binomial formula, after that, grouping in powers of h, for Y_1 we have

$$Y_1 = \sum_{\beta=0}^{t-1} C_{\beta}(x_{\beta}^{\alpha}+(1-x_{\beta})^{\alpha}) = \sum_{\beta=0}^{t-1} C_{\beta}\left(\left(\eta_{\beta}h\right)^{\alpha}+\sum_{j=0}^{\alpha}\frac{\alpha!(-\eta_{\beta}h)^j}{j!(\alpha-j)!}\right)$$
$$= h^{\alpha+1}\sum_{\beta=0}^{t-1} h^{-1}C_{\beta}\eta_{\beta}^{\alpha}(1+(-1)^{\alpha})+\sum_{j=0}^{\alpha-1}\frac{\alpha! h^{j+1}}{j!(\alpha-j)!}\sum_{\beta=0}^{t-1} h^{-1}C_{\beta}(-\eta_{\beta})^j.$$

Hence using (3.11) we obtain

$$Y_1 = h^{\alpha+1}\sum_{\beta=0}^{t-1} h^{-1}C_{\beta}\eta_{\beta}^{\alpha}(1+(-1)^{\alpha})$$
$$+\sum_{j=1}^{\alpha-1}\frac{\alpha! h^{j+1}}{j!(\alpha-j)!}\sum_{\beta=0}^{t-1} h^{-1}C_{\beta}(-\eta_{\beta})^j+h\left(\frac{2t-1}{2}-\sum_{k=1}^{m-1} d_k\frac{q_k^t-q_k^{N-t+1}}{1-q_k}\right).$$

Now, adding Y_1, Y_2 and substituting the result to the left side of equality (3.8), we obtain

$$h^{\alpha+1}\left[\sum_{k=1}^{m-1} d_k\sum_{i=1}^{\alpha}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^{\alpha}-\sum_{\beta=1}^{t-1}\beta^{\alpha}\left(1+\sum_{k=1}^{m-1} d_k\left(q_k^{\beta}+q_k^{N-\beta}\right)\right)+\sum_{\beta=0}^{t-1} h^{-1}C_{\beta}\eta_{\beta}^{\alpha}\right](1+(-1)^{\alpha})$$
$$+\sum_{j=1}^{\alpha-1}\frac{\alpha! h^{j+1}}{j!(\alpha-j)!}\left[\frac{B_{j+1}}{j+1}+\sum_{k=1}^{m-1} d_k\sum_{i=1}^{j}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^j\right.$$
$$\left.+(-1)^{j+1}\sum_{\beta=1}^{t-1}\beta^j\left(1+\sum_{k=1}^{m-1} d_k\left(q_k^{\beta}+q_k^{N-\beta}\right)\right)+\sum_{\beta=0}^{t-1} h^{-1}C_{\beta}(-\eta_{\beta})^j\right]=0. \tag{3.12}$$

It is clear that the left side of (3.12) is the polynomial of degree $\alpha+1$ with respect to h. Since this polynomial is equal to zero then all coefficients of the polynomial are zero.

When α is odd from (3.12) we get the system of equations

$$\sum_{k=1}^{m-1} d_k\left\{\sum_{i=1}^{j}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^j+(-1)^{j+1}\sum_{\beta=1}^{t-1}\beta^j\left(q_k^{\beta}+q_k^{N-\beta}\right)\right\}$$

$$+\frac{B_{j+1}}{j+1}+(-1)^j\sum_{\beta=0}^{t-1}\left(C_\beta h^{-1}\eta_\beta^j-\beta^j\right)=0,\quad j=1,2,\ldots,\alpha-1,$$

which is a part of system (4.22), because $1\le\alpha\le m-1$.

When α is even from (3.12) we get the following system of equations

$$\sum_{k=1}^{m-1}d_k\left\{\sum_{i=1}^{j}\frac{-q_k^{N+i}+(-1)^iq_k}{(1-q_k)^{i+1}}\Delta^i0^j+(-1)^{j+1}\sum_{\beta=1}^{t-1}\beta^j\left(q_k^\beta+q_k^{N-\beta}\right)\right\}$$
$$+\frac{B_{j+1}}{j+1}+(-1)^j\sum_{\beta=0}^{t-1}\left(C_\beta h^{-1}\eta_\beta^j-\beta^j\right)=0,\quad j=1,2,\ldots,\alpha-1,$$

which is a part of system (4.22) and one new equation

$$\sum_{k=1}^{m-1}d_k\left[\sum_{i=1}^{\alpha}\frac{-q_k^{N+i}+(-1)^iq_k}{(1-q_k)^{i+1}}\Delta^i0^\alpha-\sum_{\beta=1}^{t-1}\beta^\alpha\left(q_k^\beta+q_k^{N-\beta}\right)\right]+\sum_{\beta=0}^{t-1}(C_\beta h^{-1}\eta_\beta^\alpha-\beta^\alpha)=0$$

for each even α. From the last equality and (3.11) for the optimal coefficients $C_\beta,\ \beta=0,\ldots,t-1$ we get the system of equations

$$\sum_{\beta=0}^{t-1}C_\beta=h\left(\frac{2t-1}{2}-\sum_{k=1}^{m-1}d_k\frac{q_k^t-q_k^{N-t+1}}{1-q_k}\right),\tag{3.13}$$

$$\sum_{\beta=0}^{t-1}C_\beta\eta_\beta^\alpha=h\left[\sum_{\beta=1}^{t-1}\beta^\alpha-\sum_{k=1}^{m-1}d_k\left(\sum_{i=1}^{\alpha}\frac{-q_k^{N+i}+(-1)^iq_k}{(1-q_k)^{i+1}}\Delta^i0^\alpha-\sum_{\beta=1}^{t-1}\beta^\alpha\left(q_k^\beta+q_k^{N-\beta}\right)\right)\right],\tag{3.14}$$

where $\alpha=\begin{cases}2,4,\ldots,m-2 \text{ when } m- \text{ even},\\ 2,4,\ldots,m-1 \text{ when } m- \text{ odd}.\end{cases}$

Thus for the solvability of system (3.13) and (3.14) (with respect to $C_\beta,\ \beta=\overline{0,t-1}$) t has to be equal to $\frac{m}{2}$ when m is even and $\left[\frac{m}{2}\right]+1$ when m is odd.

Now, let $Y_3=\sum_{\beta=1}^{t-1}\beta^\alpha\left(q_k^\beta+q_k^{N-\beta}\right)$. Then, using formula (2.4) and Lemma 3.2 considering that α is an even natural number, we have

$$Y_3=\sum_{i=1}^{\alpha}\frac{-q_k^{N+i}+(-1)^iq_k}{(1-q_k)^{i+1}}\Delta^i0^\alpha+\sum_{i=0}^{\alpha}\frac{(-1)^iq_k^{t+i}-q_k^{N-t+1}}{(q_k-1)^{i+1}}\Delta^it^\alpha.\tag{3.15}$$

Substituting equality (3.15) into (3.14) and taking into account (3.13) we get the assertion of Lemma 3.3.

Lemma 3.3 is proved. □

Proof of Lemma 3.4. Using the formula of the binomial coefficients we get

$$\begin{aligned}C_m^lC_l^k&=\frac{m!}{l!\cdot(m-l)!}\frac{l!}{k!\cdot(l-k)!}\\&=\frac{m!}{k!\cdot(m-k)!}\cdot\frac{(m-k)!}{(l-k)!\cdot((m-k)-(l-k))!}=C_m^kC_{m-k}^{l-k}.\end{aligned}\tag{3.16}$$

Applying equality (3.16) to the left side of (3.4) we obtain

$$\begin{aligned}&C_m^{m-1}C_{m-1}^n-C_m^{m-2}C_{m-2}^n+\cdots+(-1)^{m-n-2}C_m^{n+1}C_{n+1}^n+(-1)^{m-n-1}C_m^nC_n^n\\&=C_m^nC_{m-n}^{m-n-1}-C_m^nC_{m-n}^{m-n-2}+\cdots+(-1)^{m-n-1}C_m^nC_{m-n}^0\\&=C_m^n\left(C_{m-n}^{m-n-1}-C_{m-n}^{m-n-2}+\cdots+(-1)^{m-n-1}C_{m-n}^0\right)\\&=C_m^n\left(C_{m-n}^{m-n}-\left(C_{m-n}^{m-n}-C_{m-n}^{m-n-1}+\cdots+(-1)^{m-n}C_{m-n}^0\right)\right)\\&=C_m^n\left(C_{m-n}^{m-n}-(1-1)^{m-n}\right)=C_m^n,\end{aligned}$$

Lemma 3.4 is proved. □

The identity (3.5) is well-known (see, for example, [1]). Here the proof of Lemma 3.5 is omitted.

4. The main results

4.1. Coefficients of optimal quadrature formulas

For the coefficients of optimal quadrature formulas of the form (1.1) the following theorem holds.

Theorem 4.1. *The coefficients of optimal quadrature formulas of the form* (1.1) *with the error functional* (1.2) *and the nodes* (1.5) *in the Sobolev space* $L_2^{(m)}(0,1)$ *are expressed by the formula*

$$C_\beta = h\left(1+\sum_{k=1}^{m-1} d_k\left(q_k^\beta + q_k^{N-\beta}\right)\right), \quad t \le \beta \le N-t, \tag{4.1}$$

where d_k, $(k=\overline{1,m-1})$ *satisfy the system of* $m-1$ *linear equations:*

$$\sum_{k=1}^{m-1} d_k \sum_{i=1}^{j} \frac{-q_k^{t+1}+(-1)^i q_k^{N-t+i}}{(q_k-1)^{i+1}}\Delta^i 0^j = \frac{t^{j+1}-B_{j+1}}{j+1} - \sum_{\beta=0}^{t-1} C_\beta h^{-1}(t-\eta_\beta)^j, \quad j=1,2,3,\ldots,m-1, \tag{4.2}$$

here the coefficients $C_\beta = C_{N-\beta}$ $(\beta = 0,1,\ldots,t-1)$ *are determined from system* (3.3), q_k *are roots of the Euler–Frobenius polynomial* $E_{2m-2}(q)$, $|q_k|<1$.

Proof. Let us give the main stages of the proof.

As said before optimal coefficients C_β $(\beta=\overline{0,N})$ are solutions of system (1.8), (1.9). In Lemma 3.1 we have obtained the representation of optimal coefficients C_β for $\beta=\overline{t,N-t}$. In the proof of Lemma 3.3, using the result of Lemma 3.1, we have obtained system (3.3) for optimal coefficients C_β $(\beta=\overline{0,t-1})$. Therefore, we conclude that in order to solve system (1.8), (1.9) it is sufficient to find the unknown parameters d_k and unknown polynomial $P_{m-1}(x_\beta)=\sum_{k=1}^{m-1}\lambda_\alpha x_\beta^\alpha$. For this in (1.8) substituting the expression (3.1) instead of C_β, $\beta=\overline{t,N-t}$ we will get polynomials of degree $2m$ with respect to x_β on both sides of (1.8). Equating coefficients of same powers of x_β we will get the system of $m-1$ linear equations for unknowns d_k and we will find the coefficients of the unknown polynomial $P_{m-1}(x_\beta)$. Thus the proof of the theorem will be completed.

Further we give detailed explanation of the proof of the theorem.

Now we consider equality (1.8)

$$\sum_{\gamma=0}^{N} C_\gamma \frac{|x_\beta-x_\gamma|^{2m-1}}{2\cdot(2m-1)!} + P_{m-1}(x_\beta) = \frac{x_\beta^{2m}+(1-x_\beta)^{2m}}{2\,(2m)!} \quad \text{as } x_\beta\in[0,1]. \tag{4.3}$$

We denote

$$g(x_\beta) = \sum_{\gamma=0}^{N} C_\gamma \frac{|x_\beta-x_\gamma|^{2m-1}}{2\cdot(2m-1)!}, \tag{4.4}$$

$$f(x_\beta) = \frac{x_\beta^{2m}+(1-x_\beta)^{2m}}{2\,(2m)!}. \tag{4.5}$$

Consider the function $g(x_\beta)$ and suppose $t\le\beta\le N-t$. Then

$$g(x_\beta) = \sum_{\gamma=0}^{t-1} C_\gamma \frac{(x_\beta-x_\gamma)^{2m-1}}{(2m-1)!} + \sum_{\gamma=t}^{\beta} C_\gamma \frac{(x_\beta-x_\gamma)^{2m-1}}{(2m-1)!} - \sum_{\gamma=0}^{N} C_\gamma \frac{(x_\beta-x_\gamma)^{2m-1}}{2(2m-1)!}. \tag{4.6}$$

Further we denote

$$\psi_1(x_\beta) = \sum_{\gamma=0}^{t-1} C_\gamma \frac{(x_\beta-x_\gamma)^{2m-1}}{(2m-1)!}, \tag{4.7}$$

$$\psi_2(x_\beta) = \sum_{\gamma=t}^{\beta} C_\gamma \frac{(x_\beta-x_\gamma)^{2m-1}}{(2m-1)!}, \tag{4.8}$$

$$\psi_3(x_\beta) = -\sum_{\gamma=0}^{N} C_\gamma \frac{(x_\beta-x_\gamma)^{2m-1}}{2(2m-1)!}. \tag{4.9}$$

In equality (4.8), using (3.1), adding and subtracting the following expression

$$\frac{h^{2m}}{(2m-1)!}\left(\sum_{\gamma=1}^{t-1}(\beta-\gamma)^{2m-1}+\sum_{k=1}^{m-1}d_k\left(\sum_{\gamma=1}^{t-1}q_k^{\gamma}(\beta-\gamma)^{2m-1}+\sum_{\gamma=1}^{t-1}q_k^{N-\gamma}(\beta-\gamma)^{2m-1}\right)\right),$$

we get

$$\begin{aligned}\psi_2(x_\beta)&=\frac{h^{2m}}{(2m-1)!}\left[\sum_{\gamma=1}^{\beta}(\beta-\gamma)^{2m-1}+\sum_{k=1}^{m-1}d_k\left(\sum_{\gamma=1}^{\beta}q_k^{\gamma}(\beta-\gamma)^{2m-1}+\sum_{\gamma=1}^{\beta}q_k^{N-\gamma}(\beta-\gamma)^{2m-1}\right)\right]\\&\quad-\frac{h^{2m}}{(2m-1)!}\left[\sum_{\gamma=1}^{t-1}(\beta-\gamma)^{2m-1}+\sum_{k=1}^{m-1}d_k\left(\sum_{\gamma=1}^{t-1}q_k^{\gamma}(\beta-\gamma)^{2m-1}+\sum_{\gamma=1}^{t-1}q_k^{N-\gamma}(\beta-\gamma)^{2m-1}\right)\right].\end{aligned}$$

Hence replacing $\beta-\gamma$ with γ and using equalities (2.4), (2.5)

$$\begin{aligned}\psi_2(x_\beta)&=\frac{h^{2m}}{(2m-1)!}\left[\sum_{j=1}^{2m}\frac{(2m-1)!B_{2m-j}}{j!\,(2m-j)!}\beta^j+\sum_{k=1}^{m-1}\left\{q_k^{\beta}\left(\frac{q_k}{q_k-1}\sum_{i=0}^{2m-1}\frac{\Delta^i 0^{2m-1}}{(q_k-1)^i}-\frac{q_k^{1-\beta}}{q_k-1}\sum_{i=0}^{2m-1}\frac{\Delta^i\beta^{2m-1}}{(q_k-1)^i}\right)\right.\right.\\&\quad\left.+q_k^{N-\beta}\left(\frac{1}{1-q_k}\sum_{i=0}^{2m-1}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i 0^{2m-1}-\frac{q_k^{\beta}}{1-q_k}\sum_{i=0}^{2m-1}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i\beta^{2m-1}\right)\right\}\\&\quad\left.-\sum_{\gamma=1}^{t-1}(\beta-\gamma)^{2m-1}\left(1+\sum_{k=1}^{m-1}d_k\left(q_k^{\gamma}+q_k^{N-\gamma}\right)\right)\right].\end{aligned}$$

Taking into account that q_k are roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$ and using Theorem 2.1, for $\psi_2(x_\beta)$ we have

$$\begin{aligned}\psi_2(x_\beta)&=\frac{h^{2m}}{(2m-1)!}\left[\sum_{j=1}^{2m}\frac{(2m-1)!B_{2m-j}}{j!\,(2m-j)!}\beta^j+\sum_{k=1}^{m-1}d_k\sum_{i=0}^{2m-1}\frac{-q_k+(-1)^i q_k^{N+i}}{(q_k-1)^{i+1}}\Delta^i\beta^{2m-1}\right.\\&\quad\left.-\sum_{\gamma=1}^{t-1}(\beta-\gamma)^{2m-1}\left(1+\sum_{k=1}^{m-1}d_k\left(q_k^{\gamma}+q_k^{N-\gamma}\right)\right)\right].\end{aligned}\tag{4.10}$$

Finally, using the binomial formula and (2.6), from (4.10) we get

$$\begin{aligned}\psi_2(x_\beta)&=\frac{(h\beta)^{2m}}{(2m)!}+\frac{h^{2m}\beta^{2m-1}}{(2m-1)!}\left[-\frac{2t-1}{2}+\sum_{k=1}^{m-1}d_k\frac{q_k^t-q_k^{N-t+1}}{1-q_k}\right]\\&\quad+\frac{h^{2m}}{(2m-1)!}\sum_{j=1}^{2m-2}\frac{(2m-1)!B_{j+1}\beta^{2m-1-j}}{(j+1)!(2m-1-j)!}+\frac{h^{2m}}{(2m-1)!}\sum_{j=1}^{2m-1}\frac{(2m-1)!\beta^{2m-1-j}}{j!(2m-1-j)!}\\&\quad\times\left[\sum_{k=1}^{m-1}d_k\sum_{i=1}^{j}\frac{-q_k+(-1)^i q_k^{N+i}}{(q_k-1)^{i+1}}\Delta^i 0^j-\sum_{\gamma=1}^{t-1}(-\gamma)^j\left(1+\sum_{k=1}^{m-1}d_k\left(q_k^{\gamma}+q_k^{N-\gamma}\right)\right)\right].\end{aligned}\tag{4.11}$$

Now we consider equality (4.7). Using the binomial formula, keeping in mind (1.5) and (3.3) when $\alpha=0$, for $\psi_1(x_\beta)$ we have

$$\begin{aligned}\psi_1(x_\beta)&=\sum_{\gamma=0}^{t-1}C_\gamma\frac{(x_\beta-x_\gamma)^{2m-1}}{(2m-1)!}=\sum_{\gamma=0}^{t-1}C_\gamma\frac{(h\beta-\eta_\gamma h)^{2m-1}}{(2m-1)!}\\&=\frac{h^{2m}}{(2m-1)!}\sum_{j=1}^{2m-1}\frac{(2m-1)!\beta^{2m-1-j}}{j!(2m-1-j)!}\sum_{\gamma=0}^{t-1}C_\gamma h^{-1}(-\eta_\gamma)^j\\&\quad+\frac{h^{2m}\beta^{2m-1}}{(2m-1)!}\left[\frac{2t-1}{2}-\sum_{k=1}^{m-1}d_k\frac{q_k^t-q_k^{N-t+1}}{1-q_k}\right].\end{aligned}\tag{4.12}$$

Further using the binomial formula and equality (1.9) from (4.9) we obtain

$$\psi_3(x_\beta)=-\sum_{j=0}^{m-1}\frac{(-1)^j x_\beta^{2m-1-j}}{2(j+1)!(2m-1-j)!}-\sum_{j=0}^{m-1}\frac{(-1)^{m+j}x_\beta^{m-1-j}}{2(m+j)!(m-1-j)!}\sum_{\gamma=0}^{N}C_\gamma x_\gamma^{m+j}.\tag{4.13}$$

Similarly, using the binomial formula in equality (4.5) for $f_m(x_\beta)$ we have

$$f_m(x_\beta)=\frac{x_\beta^{2m}}{(2m)!}+\sum_{j=0}^{m-1}\frac{(-1)^{j+1}x_\beta^{2m-1-j}}{2(j+1)!(2m-1-j)!}+\sum_{j=0}^{m-1}\frac{(-1)^{m+j}x_\beta^{m+1+j}}{2(m+1+j)!(m-1-j)!}. \tag{4.14}$$

Substituting equalities (4.11)–(4.14) into (4.3) and after some calculations we get

$$\frac{h^{2m}}{(2m-1)!}\sum_{j=1}^{2m-2}\frac{(2m-1)!B_{j+1}\beta^{2m-1-j}}{(j+1)!(2m-1-j)!}+\frac{h^{2m}}{(2m-1)!}\sum_{j=1}^{2m-1}\frac{(2m-1)!\beta^{2m-1-j}}{j!(2m-1-j)!}$$
$$\times\left[(-1)^j\sum_{\gamma=0}^{t-1}\left(C_\gamma h^{-1}\eta_\gamma^j-\gamma^j\right)+\sum_{k=1}^{m-1}d_k\left(\sum_{i=1}^{j}\frac{-q_k+(-1)^iq_k^{N+i}}{(q_k-1)^{i+1}}\Delta^i0^j-\sum_{\gamma=1}^{t-1}(-\gamma)^j\left(q_k^\gamma+q_k^{N-\gamma}\right)\right)\right]$$
$$+\sum_{j=0}^{m-1}\frac{(-1)^{m-j}x_\beta^{m-1-j}}{2(m+j)!(m-1-j)!}\left[\frac{1}{m+1+j}-\sum_{\gamma=0}^{N}C_\gamma x_\gamma^{m+j}\right]=-P_{m-1}(x_\beta). \tag{4.15}$$

Keeping in mind equality (4.13) and designations (4.4) and (4.5), from (4.15) one can see that the difference $g(x_\beta)-f_m(x_\beta)$ is a polynomial of degree $2m-2$ with respect to x_β, i.e.

$$g(x_\beta)-f_m(x_\beta)=\sum_{j=0}^{2m-2}a_jx_\beta^j,\quad t\le\beta\le N-t. \tag{4.16}$$

Here

$$a_j=\begin{cases} b_j \quad \text{as } m\le j\le 2m-2,\\ b_j+\dfrac{(-1)^{j+1}}{2j!(2m-j-1)!}\left[\dfrac{1}{2m-j}-\displaystyle\sum_{\gamma=0}^{N}C_\gamma x_\gamma^{2m-j-1}\right] \quad \text{as } 1\le j\le m-1,\\ \dfrac{h^{2m}}{(2m-1)!}\left[(-1)^{2m-1}\displaystyle\sum_{\gamma=0}^{t-1}\left(C_\gamma h^{-1}\eta_\gamma^{2m-1}-\gamma^{2m-1}\right)\right.\\ \quad\left.+\displaystyle\sum_{k=1}^{m-1}d_k\left(\sum_{i=1}^{2m-1}\frac{-q_k+(-1)^iq_k^{N+i}}{(q_k-1)^{i+1}}\Delta^i0^{2m-1}+\sum_{\gamma=1}^{t-1}\gamma^{2m-1}\left(q_k^\gamma+q_k^{N-\gamma}\right)\right)\right]\\ \quad-\dfrac{1}{2(2m-1)!}\left(\dfrac{1}{2m}-\displaystyle\sum_{\gamma=0}^{N}C_\gamma x_\gamma^{2m-1}\right) \quad \text{as } j=0, \end{cases} \tag{4.17}$$

where

$$b_j=\frac{h^{2m-j}}{j!(2m-j-1)!}\left[\frac{B_{2m-j}}{2m-j}+(-1)^{2m-j-1}\sum_{\gamma=0}^{t-1}\left(C_\gamma h^{-1}\eta_\gamma^{2m-j-1}-\gamma^{2m-j-1}\right)\right.$$
$$\left.+\sum_{k=1}^{m-1}d_k\left(\sum_{i=1}^{2m-j-1}\frac{-q_k+(-1)^iq_k^{N+i}}{(q_k-1)^{i+1}}\Delta^i0^{2m-1-j}+(-1)^{2m-j}\sum_{\gamma=1}^{t-1}\gamma^{2m-j-1}\left(q_k^\gamma+q_k^{N-\gamma}\right)\right)\right].$$

On the other hand, from (4.15) we obtain

$$g(x_\beta)-f_m(x_\beta)=-P_{m-1}(x_\beta). \tag{4.18}$$

This equality is true if $b_j=0$ as $m\le j\le 2m-2$ or

$$\sum_{k=1}^{m-1}d_k\left\{\sum_{i=1}^{j}\frac{-q_k+(-1)^iq_k^{N+i}}{(q_k-1)^{i+1}}\Delta^i0^j-(-1)^j\sum_{\gamma=1}^{t-1}\gamma^j\left(q_k^\gamma+q_k^{N-\gamma}\right)\right\}$$
$$=-\frac{B_{j+1}}{j+1}-(-1)^j\sum_{\gamma=0}^{t-1}\left(C_\gamma h^{-1}\eta_\gamma^j-\gamma^j\right),\quad j=1,2,\dots,m-1. \tag{4.19}$$

From equalities (4.16) and (4.18) we find an unknown polynomial $P_{m-1}(x_\beta)$ of system (1.8), (1.9)

$$P_{m-1}(x_\beta) = -\sum_{j=0}^{m-1} a_j x_\beta^j. \tag{4.20}$$

Later, applying to the sum $A = \sum_{\gamma=1}^{t-1} \gamma^j \left(q_k^\gamma + q_k^{N-\gamma}\right)$ formulas (2.4)–(2.6) and (3.2), we obtain

$$A = (-1)^j \sum_{i=1}^{j} \frac{-q_k + (-1)^i q_k^{N+i}}{(q_k-1)^{i+1}} \Delta^i 0^j + \sum_{i=0}^{j} \frac{-q_k^{N-t+1} + (-1)^i q_k^{t+i}}{(q_k-1)^{i+1}} \sum_{p=0}^{j} C_j^p \Delta^i 0^p t^{j-p}. \tag{4.21}$$

Substituting equality (4.21) into (4.19), after some simplifications we have

$$\sum_{k=1}^{m-1} d_k \left\{ (-1)^{j+1} \sum_{p=0}^{j} C_j^p t^{j-p} \sum_{i=0}^{p} \frac{-q_k^{N-t+1} + (-1)^i q_k^{t+i}}{(q_k-1)^{i+1}} \Delta^i 0^p \right\}$$
$$= -\frac{B_{j+1}}{j+1} - (-1)^j \sum_{\gamma=0}^{t-1} \left(C_\gamma h^{-1} \eta_\gamma^j - \gamma^j\right), \quad j = 1, 2, \ldots, m-1. \tag{4.22}$$

Multiplying the first equation of system (4.22) by $(-1)^2 C_2^1 t$, adding with the second one, then, multiplying the first equation by $(-1)^3 C_3^1 t^2$, the second by $(-1)^2 C_3^2 t$ of system (4.22), adding with the third and so on, continuing in this way, and also taking into account Lemmas 3.2–3.5 and the binomial formula for unknown parameters d_k we get the following linear system of equations

$$\sum_{k=1}^{N} d_k \sum_{i=1}^{j} \frac{-q_k^{t+1} + (-1)^i q_k^{N-t+i}}{(q_k-1)^{i+1}} \Delta^i 0^j = \frac{t^{j+1} - B_{j+1}}{j+1} - \sum_{\beta=0}^{t-1} C_\beta h^{-1} (t - \eta_\beta)^j, \quad j = 1, 2, \ldots, m-1.$$

The last system is system (4.2) for d_k. This completes the proof of Theorem 4.1. □

4.2. The norm of the error functional of the optimal quadrature formula

In this section, square of the norm of the error functional (1.2) of optimal quadrature formulas (1.1) with the nodes (1.5) is calculated.

Theorem 4.2. *Square of the norm of the error functional (1.2) of optimal quadrature formulas (1.1) with the nodes (1.5) on the space $L_2^{(m)}(0,1)$ have the form*

$$\left\| \overset{\circ}{\ell}(x) | L_2^{(m)*}(0,1) \right\|^2 = (-1)^{m+1} \left[\frac{h^{2m} B_{2m}}{(2m)!} + \frac{2h^{2m+1}}{(2m)!} \left\{ \sum_{\beta=0}^{t-1} \left(C_\beta h^{-1} \eta_\beta^{2m} - \beta^{2m}\right) \right.\right.$$
$$\left.\left. + \sum_{k=1}^{m-1} d_k \sum_{i=0}^{2m} \frac{(-1)^i q_k^{t+i} - q_k^{N-t+1}}{(q_k-1)^{i+1}} \Delta^i t^{2m} \right\} \right],$$

where B_α are Bernoulli numbers, $C_\beta, \beta = \overline{0, t-1}$ are determined from system (3.3), q_k are roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$, $|q_k| < 1$, $\eta_\beta, \beta = \overline{0, t-1}$ are defined from (1.5), $\Delta^i t^{2m}$ is the finite difference of order i of t^{2m}, d_k are determined from (4.2).

Proof. Taking into account (4.3), we reduce the expression (1.7) to the form

$$\left\| \ell(x) | L_2^{(m)*}(0,1) \right\|^2 = (-1)^{m+1} \left[\sum_{\beta=0}^{N} C_\beta \left(f_m(x_\beta) + P_{m-1}(x_\beta)\right) - \frac{1}{(2m+1)!} \right]. \tag{4.23}$$

Applying the binomial formula to equality (4.5), we obtain

$$f_m(x_\beta) = \frac{(1-x_\beta)^{2m}}{(2m)!} - \sum_{i=0}^{2m-1} \frac{(-1)^{i+1} x_\beta^{2m-i-1}}{2(i+1)!(2m-i-1)!}. \tag{4.24}$$

Substituting (4.24) into (4.23) and using equalities (4.17) and (1.9), consecutively we have

$$\begin{aligned}\|\ell(x)\|^2 &= (-1)^{m+1}\Bigg[\sum_{\beta=0}^{N} C_\beta \frac{(1-x_\beta)^{2m}}{(2m)!} - \frac{1}{(2m+1)!} - \sum_{j=1}^{m-1}\frac{h^{2m-j}B_{2m-j}}{(j+1)!(2m-j)!}\\ &\quad - \sum_{j=0}^{m-1}\frac{h^{2m-j}}{(j+1)!(2m-j-1)!}\Bigg\{(-1)^{2m-j-1}\sum_{\gamma=0}^{t-1}\left(C_\gamma h^{-1}\eta_\gamma^{2m-j-1}-\gamma^{2m-j-1}\right)\\ &\quad + \sum_{k=1}^{m-1} d_k\left(\sum_{i=1}^{2m-j-1}\frac{-q_k+(-1)^i q_k^{N+i}}{(q_k-1)^{i+1}}\Delta^i 0^j + (-1)^{2m-j}\sum_{\gamma=1}^{t-1}\gamma^{2m-j-1}\left(q_k^\gamma + q_k^{N-\gamma}\right)\right)\Bigg\}\Bigg]\\ &= (-1)^{m+1}\left[Z_1+Z_2\right],\end{aligned} \tag{4.25}$$

where $Z_1 = \sum_{\beta=0}^{N} C_\beta \frac{(1-x_\beta)^{2m}}{(2m)!}$ and Z_2 is the remaining part in square brackets of equality (4.25).

Keeping in mind the symmetry of the nodes (1.5), and making the same calculations as in the proof of Lemma 3.3 when $\alpha = 2m$, for Z_1 we have

$$\begin{aligned} Z_1 &= \frac{1}{(2m+1)!} + \frac{h^{2m}B_{2m}}{(2m)!} + \sum_{j=1}^{2m-2}\frac{B_{2m-j}h^{2m-j}}{(j+1)!(2m-j)!}\\ &\quad + \sum_{j=0}^{2m-2}\frac{h^{2m-j}}{(j+1)!(2m-j-1)!}\Bigg[(-1)^{2m-j-1}\sum_{\beta=0}^{t-1}\left(C_\beta h^{-1}\eta_\beta^{2m-j-1}-\beta^{2m-j-1}\right)\\ &\quad + \sum_{k=1}^{m-1} d_k\left\{\sum_{i=1}^{2m-j-1}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^{2m-j-1} + (-1)^{2m-j}\sum_{\beta=0}^{t-1}\beta^{2m-j-1}\left(q_k^\beta+q_k^{N-\beta}\right)\right\}\Bigg]\\ &\quad + \frac{2h^{2m+1}}{(2m)!}\left[\sum_{k=1}^{m-1} d_k\left\{\sum_{i=1}^{2m}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^{2m} - \sum_{\beta=0}^{t-1}\beta^{2m}\left(q_k^\beta+q_k^{N-\beta}\right)\right\} + \sum_{\beta=0}^{t-1}\left(C_\beta h^{-1}\eta_\beta^{2m}-\beta^{2m}\right)\right].\end{aligned} \tag{4.26}$$

Putting equality (4.26) into (4.25) and taking into account equality (4.19), we have

$$\begin{aligned}\|\ell(x)\|^2 &= (-1)^{m+1}\Bigg[\frac{h^{2m}B_{2m}}{(2m)!} + \frac{2h^{2m+1}}{(2m)!}\Bigg[\sum_{k=1}^{m-1} d_k\left\{\sum_{i=1}^{2m}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^{2m} - \sum_{\beta=1}^{t-1}\beta^{2m}\left(q_k^\beta+q_k^{N-\beta}\right)\right\}\\ &\quad + \sum_{\beta=0}^{t-1}\left(C_\beta h^{-1}\eta_\beta^{2m}-\beta^{2m}\right)\Bigg]\Bigg].\end{aligned} \tag{4.27}$$

Hence, using formulas (2.4), (2.5) and Lemma 3.2 when $s = 0$, after some simplifications we get the assertion of the theorem.

Theorem 4.2 is proved. □

Remark. It should be noted that when the nodes (1.5) are equally spaced, i.e. when in (1.5) $\eta_0 = 0$, $\eta_1 = 1, \ldots, \eta_{t-1} = t-1$, from Theorems 4.1 and 4.2 we get Theorem 2.1 of [17] and the results of [29].

5. Numerical results

For simplicity, in this section we investigate the optimal quadrature formulas of the form (1.1) with the nodes

$$x_0 = \eta_0 h, \qquad x_N = 1-\eta_0 h, \quad 0 \le \eta_0 < 1, \qquad x_\beta = h\beta, \quad \beta = 1, 2, \ldots, N-1. \tag{5.1}$$

This means that we consider optimal quadrature formulas of the form

$$\int_0^1 \varphi(x)dx \cong C_0\varphi(\eta_0 h) + \sum_{\beta=1}^{N-1} C_\beta\varphi(h\beta) + C_N\varphi(1-\eta_0 h) \tag{5.2}$$

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \left(C_0\delta(x-\eta_0 h) + \sum_{\beta=1}^{N-1} C_\beta\delta(x-h\beta) + C_N\delta(x-(1-\eta_0 h))\right). \tag{5.3}$$

Then for this case from Theorems 4.1 and 4.2 when $t = 1$ we get the following

Corollary 5.1. *The coefficients of optimal quadrature formulas (5.2) with the error functional (5.3) and the nodes (5.1) in the Sobolev space $L_2^{(m)}(0,1)$ are expressed by formulas*

$$C_\beta = \begin{cases} h\left(\frac{1}{2}+\sum_{k=1}^{m-1} d_k \frac{q_k - q_k^N}{q_k-1}\right), & \beta = 0,\ N, \\ h\left(1+\sum_{k=1}^{m-1} d_k \left(q_k^\beta + q_k^{N-\beta}\right)\right), & \beta = 1,2,\dots,N-1, \end{cases} \tag{5.4}$$

where d_k ($k=\overline{1,m-1}$) satisfy the following system of $m-1$ linear equations

$$\sum_{k=1}^{m-1} d_k \left(\sum_{i=1}^{j} \frac{-q_k^2+(-1)^i q_k^{N-1+i}}{(q_k-1)^{i+1}}\Delta^i 0^j + \frac{(q_k-q_k^N)(1-\eta_0)^j}{q_k-1}\right) = \frac{1-B_{j+1}}{j+1} - \frac{(1-\eta_0)^j}{2}, \quad j=1,2,\dots,m-1, \tag{5.5}$$

here q_k are roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$ of degree $2m-2$, $|q_k|<1$, B_{j+1} are Bernoulli numbers, $0\le\eta_0<1$, $\Delta^i\gamma^j$ is finite difference of order i of γ^j, $\Delta^i 0^j = \Delta^i\gamma^j|_{\gamma=0}$.

Corollary 5.2. *Square of the norm of the error functional (5.3) of optimal quadrature formulas (5.2) on the space $L_2^{(m)}(0,1)$ have the form*

$$\left\|\overset{\circ}{\ell}(x)|L_2^{(m)*}(0,1)\right\|^2$$
$$= (-1)^{m+1}\left[\frac{h^{2m}B_{2m}}{(2m)!} + \frac{2h^{2m+1}}{(2m)!}\left(\frac{\eta_0^{2m}}{2} + \sum_{k=1}^{m-1} d_k\left[\sum_{i=1}^{2m}\frac{-q_k^{N+i}+(-1)^i q_k}{(1-q_k)^{i+1}}\Delta^i 0^{2m} + \frac{(q_k-q_k^N)\eta_0^{2m}}{q_k-1}\right]\right)\right], \tag{5.6}$$

where B_{2m} is the Bernoulli number, q_k are roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$, $|q_k|<1$, $0\le\eta_0<1$, $\Delta^i\gamma^{2m}$ is the finite difference of order i of γ^{2m}, $\Delta^i 0^{2m} = \Delta^i\gamma^{2m}|_{\gamma=0}$.

Clearly there is one free parameter η_0 in system (5.5). The aim of this section is to investigate the positiveness of optimal coefficients using equalities (5.4) and (5.5) and to choose a free parameter η_0, where $0\le\eta_0<1$.

As mentioned above, Schoenberg and Silliman in [26] showed that in the case of equal spaced nodes, among optimal coefficients, a negative coefficient starting from $m=7$ appears. From (5.4) and (5.5) in the case $\eta_0=0$ and $N\to\infty$ taking instead of optimal coefficients C_β the coefficients NC_β we get the results of [26].

By choosing the value of $\eta_0=0.205$ and using (5.4) and (5.5), we have obtained optimal quadrature formulas of the form (5.2) with positive coefficients in the cases $m=2,3,\dots,14$, $N=300$. These numerical results are obtained by using the Maple program which is given in Section 6. Note that when $m\ge 15$, $\eta_0=0.205$, $N=300$ the negative coefficients appear in the quadrature formula (5.2).

Now we compare some of the results of this work with the example (e) of [18].

Assume that $m=2$ and the nodes (5.1) are equally spaced, i.e. in (5.1) $\eta_0=0$. Then using (5.4) and (5.5) we get optimal quadrature formulas with equally spaced nodes in the space $L_2^{(2)}(0,1)$ for an arbitrary natural number N. These optimal quadrature formulas for $N\le 18$ were already obtained by Sard in [20,32]. From (5.6) when $N=2,3,4,5$ for the remainders $|R[\varphi]| = |(\ell(x),\varphi(x))|$ of these optimal formulas we get the following estimations:

$$\begin{aligned} N=2: |R[\varphi]| &\le \|\varphi|L_2^{(2)}(0,1)\|\cdot 0.01398,\\ N=3: |R[\varphi]| &\le \|\varphi|L_2^{(2)}(0,1)\|\cdot 0.00586,\\ N=4: |R[\varphi]| &\le \|\varphi|L_2^{(2)}(0,1)\|\cdot 0.00305,\\ N=5: |R[\varphi]| &\le \|\varphi|L_2^{(2)}(0,1)\|\cdot 0.00188. \end{aligned} \tag{5.7}$$

It should be noted, that in the example (e) of the work [18] the estimations (5.7) were obtained using the φ-function method.

Now we consider the case $m=2$ with the nodes (5.1) when $\eta_0=0.205$. Then using (5.4) and (5.5) we obtain optimal quadrature formulas of the form (5.2) in the space $L_2^{(2)}(0,1)$ for any natural number N. For the remainders $|R[\varphi]| = |(\ell(x),\varphi(x))|$ of these optimal formulas from (5.6) when $N=2,3,4,5$ we get the following estimations:

$$\begin{aligned} N=2: |R[\varphi]| &\le \|\varphi|L_2^{(2)}(0,1)\|\cdot 0.00694814,\\ N=3: |R[\varphi]| &\le \|\varphi|L_2^{(2)}(0,1)\|\cdot 0.00340515,\\ N=4: |R[\varphi]| &\le \|\varphi|L_2^{(2)}(0,1)\|\cdot 0.00203429,\\ N=5: |R[\varphi]| &\le \|\varphi|L_2^{(2)}(0,1)\|\cdot 0.00134083. \end{aligned} \tag{5.8}$$

Thus it is clear that in the space $L_2^{(2)}(0,1)$ for $N=2,3,4,5$ the errors (5.8) of the optimal quadrature formulas of the form (5.2) with the nodes (5.1) when $\eta_0=0.205$ are less than the errors (5.7) of the optimal quadrature formulas with equally spaced nodes.

Note that one can easily get these numerical results using the maple program which is given in Section 6.

6. The maple program for computation of the optimal coefficients in the space $L_2^{(m)}(0,1)$

In this section we give the Maple program for the computation of the coefficients (5.4) and the norm of the error functional (5.6) of optimal quadrature formulas (5.2) with the nodes (5.1) in the sense of Sard in the space $L_2^{(m)}(0,1)$ for any natural numbers m and N, $N\geq m$. In the program below the case $m=2$, $N=300$, $\eta_0=0.205$ is given. Changing the values of m, N and $0\leq\eta_0<1$ we can get new optimal quadrature formulas.

```
>restart;
>with(LinearAlgebra); Digits:= 100; m:= 2; N:= 300; eta0:= 0.205; h:= 1/N;
```

Computation of coefficients of the Euler–Frobenius polynomial $E_{2m-2}(x)$ of degree $2m-2$. Here for the coefficients e_s, $s=\overline{0,2m-2}$ of the polynomial $E_{2m-2}(x)=\sum_{s=0}^{2m-2}e_sx^s$ the formula $e_s=\sum_{j=0}^{s}(-1)^j\binom{2m}{j}(s+1-j)^{2m-1}$ is used which was given by Euler

```
>e:= array(0..2*m-2);
> for s from 0 to 2*m-2 do
    e[s]:= sum((-1)^j*binomial(2*m,j)*(s+1-j)^(2*m-1),j=0..s)
  end do;
```

The explicit form of the Euler–Frobenius polynomial $E_{2m-2}(x)$ of degree $2m-2$.

```
> E:= sum(e[i]*x^i,i=0..2*m-2);
```

Computation of roots q_k, $k=1,2,\ldots,2m-2$ of the polynomial $E_{2m-2}(x)$

```
> p:= solve(E,x);
```

The list of roots q_k, $k=1,2,\ldots,m-1$ of the Euler–Frobenius polynomial of degree $2m-2$, where $|q_k|<1$

```
> q:= array(1..m-1);
> for i from 1 to m-1 do q[i]:= evalf(p[i]) end do;
> print(q);
```

Computation the quantities Δ^i0^j, $i=\overline{1,2m}$, $j=\overline{1,2m}$

```
> delta:= array(1..2*m,1..2*m);
> for j from 1 to 2*m do
  for i from 1 to 2*m do
     delta[j,i]:= sum((-1)^(i-l)*binomial(i,l)*l^j,l=0..i);\\
  end do;
  end do;
> print(delta);
```

Computation of the matrix A and the vector B. Here A is the main matrix and B is the vector on the right hand side of system (5.5)

```
> A:= Matrix(1..m-1,1..m-1); B:= Vector(1..m-1);
> for j1 from 1 to m-1 do
  for k  from 1 to m-1 do
   A[j1,k]:= sum((-q[k]^{2}+(-1)^i1*q[k]^(N-1+i1))/(q[k]-1)^(i1+1)*
     delta[j1,i1],i1=1..j1)+(q[k]-q[k]^N)*(1-eta0)^j1/(q[k]-1);
   B[j1]:= (1-bernoulli(j1+1))/(j1+1)-(1-eta0)^j1/2;
  end do;
  end do;
```

Using LU decomposition method here system (5.5) is solved

```
> d:= LinearSolve(A,B, method='LU');
> d1:= array(1..m-1);
> for i from 1 to m-1 do
   d1[i]:= d[i];
  end do;
```

Using the solution d_k, $k = 1, 2, \ldots, m-1$ of system (5.5) the optimal coefficients C_β, $\beta = 0, 1, \ldots, N$ are calculated and the number of negative optimal coefficients is counted

```
> C:= array(0..N);
> negative:= 0;
>    C[0]:= h*(1/2+sum(d1[l]*(q[l]-q[l]^N)/(q[l]-1),l=1..m-1));
     if C[0]<0 then negative:= negative+1 end if;
   for beta from 1 to N-1 do
     C[beta]:= h*(1+sum(d1[l]*(q[l]^beta+q[l]^(N-beta)),l=1..m-1));
     if C[beta]<0 then negative:= negative+1 end if;
   end do;
     C[N]:= h*(1/2+sum(d1[l]*(q[l]-q[l]^N)/(q[l]-1),l=1..m-1));
     if C[N]<0 then negative:= negative+1 end if;
```

The number of negative optimal coefficients

```
> negative;
```

Calculation of the norm of the error functional (5.3), using (5.6)

```
>Norma:= 0;
> for t from 1 to m-1 do
Norma:= Norma
+d1[t]*(sum(((-1)^alpha*q[t]-q[t]^(N+alpha))/(1-q[t])^(alpha+1)
*delta[2*m,alpha],alpha=1..2*m)+(q[t]-q[t]^N)*eta0^(2*m)/(q[t]-1));
end do;
> Norma:= sqrt((-1)^(m+1)*(h^(2*m)*bernoulli(2*m)/(2*m)!+
(Norma+eta0^(2*m)/2)*2*h^(2*m+1)/(2*m)!));
```

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